

Regular variation of infinite series of processes with random coefficients

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Abstract

In this article, we consider a series $X(t) = \sum_{j \geq 1} \Psi_j(t)Z_j(t)$, $t \in [0, 1]$ of random processes with sample paths in the space \mathbb{D} of càdlàg functions (i.e. right-continuous functions with left limits) on $[0, 1]$. We assume that $(Z_j)_{j \geq 1}$ are i.i.d. processes with sample paths in \mathbb{D} and $(\Psi_j)_{j \geq 1}$ are processes with continuous sample paths. Using the notion of regular variation for \mathbb{D} -valued random elements (introduced in [13]), we show that X is regularly varying if Z_1 is regularly varying, $(\Psi_j)_{j \geq 1}$ satisfy some moment conditions, and a certain “predictability assumption” holds for the sequence $\{(Z_j, \Psi_j)\}_{j \geq 1}$. Our result can be viewed as an extension of Theorem 3.1 of [15] from random vectors in \mathbb{R}^d to random elements in \mathbb{D} . As a preliminary result, we prove a version of Breiman’s lemma for \mathbb{D} -valued random elements, which can be of independent interest.

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1 Introduction

Regular variation is an important property which lies at the core of some fundamental results in probability theory, which describe the asymptotic behavior of the maximum and the sum of n i.i.d. random variables. In the past 30 years, especially after the publication of the landmark articles [9, 21] and monograph [22], this has become a very active area of research, with a huge potential for applications, arising usually in the context of time series models.

A random variable Z is *regularly varying* if $P(|Z| > x) = x^{-\alpha}L(x)$ for any $x > 0$, and $P(Z > x)/P(|Z| > x) \rightarrow p$ as $x \rightarrow \infty$, for some $\alpha > 0, p \in [0, 1]$ and a slowly varying function L . This is equivalent to the fact that $|Z|$ is in the maximal domain of attraction of the Fréchet distribution $\Phi_\alpha(x) = \exp(-x^{-\alpha})$, and if $\alpha < 2$, to the fact that Z is in the domain of attraction of a stable distribution with index α . Another useful characterization states that Z is regularly varying if and only if there exists a sequence $(a_n)_n \uparrow \infty$ such that $nP(|Z| > a_n x) \rightarrow x^{-\alpha}$ for any $x > 0$. By considering the state space $\overline{\mathbb{R}}_0 = [-\infty, \infty] \setminus \{0\}$ instead of \mathbb{R} (such that sets of the form $[-\infty, -x) \cup (x, \infty], x > 0$ become relatively compact), the previous convergence can be expressed as the vague convergence of Radon measures:

$$nP(a_n^{-1}Z \in \cdot) \xrightarrow{v} \nu(\cdot) \quad \text{in } \overline{\mathbb{R}}_0,$$

where $\nu(dx) = [p\alpha x^{-\alpha-1}1_{(0,\infty)}(x) + (1-p)\alpha(-x)^{-\alpha-1}1_{(-\infty,0)}(x)]dx$ is a measure on $\overline{\mathbb{R}}_0$ with $\nu(\overline{\mathbb{R}}_0 \setminus \mathbb{R}) = 0$ (see e.g. Section 3.6 of [23]).

Recall that a measure μ on a locally compact space with a countable basis (LCCB) is *Radon* if $\mu(B) < \infty$ for any relatively compact Borel set B . A sequence $(\mu_n)_n$ of Radon measures *converges vaguely* to a Radon measure μ (written as $\mu_n \xrightarrow{v} \mu$) if $\mu_n(B) \rightarrow \mu(B)$ for any relatively compact Borel set B with $\mu(\partial B) = 0$ (see Appendix 15.7 of [16]).

In higher dimensions, a random vector \mathbf{Z} with values in \mathbb{R}^d is called *regularly varying* if there exist a sequence $(a_n)_n \uparrow \infty$ and a non-null Radon measure ν on $\overline{\mathbb{R}}_0^d = [-\infty, \infty]^d \setminus \{\mathbf{0}\}$ such that $\nu(\overline{\mathbb{R}}_0^d \setminus \mathbb{R}^d) = 0$ and

$$nP(a_n^{-1}\mathbf{Z} \in \cdot) \xrightarrow{v} \nu(\cdot) \quad \text{in } \overline{\mathbb{R}}_0^d. \tag{1}$$

In this case, we write $Z \in \text{RV}(\{\mathbf{a}_n\}, \nu, \overline{\mathbb{R}}_0^d)$. It can be proved that the measure ν satisfies the following *scaling property*: there exists $\alpha > 0$ (called the index

of \mathbf{Z}) such that $\nu(sB) = s^{-\alpha}\nu(B)$ for any $s > 0$ and for any Borel set $B \subset \overline{\mathbb{R}}_0^d$. In particular, for any norm $\|\cdot\|$ on \mathbb{R}^d and for any $r > 0$,

$$nP(\|\mathbf{Z}\| > a_n r) \rightarrow cr^{-\alpha},$$

where $c = \nu(\{\mathbf{x} \in \overline{\mathbb{R}}_0^d; \|\mathbf{x}\| > 1\})$. Let $\mathbb{R}_0^d = \mathbb{R}^d \setminus \{\mathbf{0}\}$, $\mathbb{S} = \{\mathbf{x} \in \mathbb{R}^d; \|\mathbf{x}\| = 1\}$ be the unit sphere in \mathbb{R}^d and $T : \mathbb{R}_0^d \rightarrow (0, \infty) \times \mathbb{S}$ be the polar coordinate transformation: $T(\mathbf{x}) = (\|\mathbf{x}\|, \mathbf{x}/\|\mathbf{x}\|)$. The fact that $\mathbf{Z} \in \text{RV}(\{a_n\}, \nu, \overline{\mathbb{R}}_0^d)$ can also be expressed as: (see e.g. Section 6.1 of [23])

$$nP(T(a_n^{-1}\mathbf{Z}) \in \cdot) \xrightarrow{v} c\nu_\alpha \times \sigma \quad \text{in } (0, \infty] \times \mathbb{S} \quad (2)$$

where $\nu_\alpha(r, \infty) = r^{-\alpha}$ and σ is a probability measure on \mathbb{S} given by $\sigma(S) = c^{-1}\nu(\{\mathbf{x} \in \overline{\mathbb{R}}_0^d; \|\mathbf{x}\| > 1, \mathbf{x}/\|\mathbf{x}\| \in S\})$. Hence, $\nu \circ T^{-1} = c\nu_\alpha \times \sigma$ on $(0, \infty) \times \mathbb{S}$.

In the one dimensional case, many time series models can be expressed as linear processes of the form:

$$X_i = \sum_{j \geq 0} c_j Z_{i-j}, \quad i \in \mathbb{Z} \quad (3)$$

where $(c_j)_{j \geq 1}$ are real numbers and $(Z_j)_{j \in \mathbb{Z}}$ are i.i.d. random variables. One simple example is the auto-regressive model of order 1, $X_i = aX_{i-1} + Z_i$ with $|a| < 1$, leading to $X_i = \sum_{j \geq 0} a^j Z_{i-j}$. Assume that Z_0 is regularly varying with index α and slowly varying function L as above. A basic question is: under what conditions the series (3) converges and if so, is X_0 still regularly varying? If $\alpha < 2$, an argument which can be traced back to [1] (see also Proposition 2.1 of [2]) shows that the series (3) converges a.s. if and only if

$$\sum_{j \geq 0} |c_j|^\alpha L(1/|c_j|) < \infty.$$

In [19], it was shown that sufficient conditions for the converges of the series (3) are: $\sum_{j \geq 1} |c_j|^{\alpha-\gamma} < \infty$ for some $\gamma \in (0, \alpha)$, if $\alpha \leq 2$, and $\sum_{j \geq 0} |c_j|^2 < \infty$ if $\alpha > 2$, and under these conditions, X_0 is regularly varying. This result continues to hold for d -dimensional vectors $(Z_j)_j$, and deterministic $p \times d$ matrices $(A_j)_j$ replacing the coefficients $(c_j)_j$ (see Corollary 3.1 of [15]).

More interesting models lead to series of the form:

$$X_i = \sum_{j \geq 0} C_{i,j} Z_{i-j}, \quad i \in \mathbb{Z} \quad (4)$$

with random coefficients $C_{i,j}$. One such example is the stochastic recurrence equation (SRE) $X_i = Y_i X_{i-1} + Z_i$, where $\{(Y_i, Z_i)\}_{i \in \mathbb{Z}}$ is a sequence of i.i.d. random vectors in \mathbb{R}^2 . Under certain conditions, it can be shown that (SRE) has a unique stationary solution which can be represented as a series of the form (4) with $C_{i,0} = 1$ and $C_{i,j} = \prod_{k=1}^j Y_{i-k+1}$ for $j \geq 1$. Another example is the bilinear model $X_i = c X_{i-1} Z_{i-1} + Z_i$, which admits a stationary solution of the form $X_i = Z_i + \sum_{j \geq 1} C_{i,j} Z_{i-j}^2$ with $C_{i,j} = \prod_{k=1}^{j-1} Z_{i-k}$. Assume that Z_0 is regularly varying. The first result addressing the question mentioned above is due to [24], where it was shown that a series X_0 given by (4) is regularly varying, if $(C_{0,j})_{j \geq 0}$ and $(Z_{-j})_{j \geq 0}$ are independent and $(C_{0,j})_{j \geq 0}$ satisfy some moment conditions. This result was extended to (SRE) and the bilinear model in [11], respectively [10]. This was improved in [15], under weaker moment conditions on $(C_{0,j})_{j \geq 0}$ and a certain “predictability assumption” imposed on the sequence $\{(C_{0,j}, Z_{-j})_{j \geq 0}\}$. The result of [15] is in fact valid for random vectors, one of the major applications of (SRE) in higher dimensions being the GARCH model (see [3]).

A breakthrough idea, which gave a new perspective to the concept of regular variation and lead to a different line of investigations, was introduced in [12]. Motivated by extreme value theory, this idea was to examine the *global* asymptotic behavior (as t runs in a fixed interval $[0, 1]$) of the normalized maximum process $\{a_n^{-1} \max_{1 \leq i \leq n} Z_i(t)\}_{t \in [0,1]}$ associated with n i.i.d. processes Z_1, \dots, Z_n whose sample paths are càdlàg functions (i.e. continuous functions with left limits) on $[0, 1]$. Each process Z_i is interpreted as a collection of measurements observed continuously over a fixed linear spatial region, $Z_i(t)$ being the observation at location t and time i . In the example of [12], $Z_i(t)$ is the high tide water level at location t and time i , along the northern coast of the Netherlands. It turns out that if Z_0 is regularly varying (in a sense which is made precise in Section 2 below), then the finite-dimensional distributions of the normalized maximum process converge to those of a càdlàg process $Y = \{Y(t)\}_{t \in [0,1]}$, and if Y has continuous sample paths, then the convergence is in distribution in the space $\mathbb{D} = \mathbb{D}[0, 1]$ of càdlàg functions on $[0, 1]$, endowed with the Skorohod J_1 -topology (see Theorem 2.4 of [12]).

The notion of regular variation for càdlàg processes was thoroughly studied in [13] where it was proved that it is equivalent to the regular variation of the finite-dimensional distributions of the process, combined with some relative compactness conditions (see Theorem 10 of [13]). In particular, a Lévy process $\{Z(t)\}_{t \in [0,1]}$ is regularly varying if and only if $Z(1)$ is regularly

varying (see Lemma 2.1 of [14]). One example is the α -stable Lévy process.

In this context, it becomes interesting to examine the regular variation of \mathbb{D} -valued time series of the form:

$$X_i(t) = \sum_{j \geq 0} \psi_j(t) Z_{i-j}(t), \quad t \in [0, 1], i \in \mathbb{Z} \quad (5)$$

where $Z_i = \{Z_i(t)\}_{t \in [0, 1]}$, $i \in \mathbb{Z}$ are i.i.d. regularly varying processes and $\psi_j = \{\psi_j(t)\}_{t \in [0, 1]}$, $j \geq 0$ are deterministic functions in \mathbb{D} . This analysis was carried out in [8], where it was proved that, under some conditions on the coefficients $(\psi_j)_{j \geq 0}$, the series X_0 given by (5) converges a.s. and is regularly varying in \mathbb{D} . Moreover, the authors of [8] derived the limit distribution of the normalized space-time maximum $a_n^{-1} \max_{i \leq n} \sup_{t \in [0, 1]} |X_i(t)|$, using non-trivial point process techniques. We should note that the results of [8] are in fact valid for càdlàg processes indexed by $[0, 1]^d$ with $d \geq 1$, being motivated by applications to spatial processes.

In the present article, we consider the next natural step in this line of investigations which consists in examining series of the form:

$$X_i(t) = \sum_{j \geq 0} \Psi_{i,j}(t) Z_{i-j}(t), \quad t \in [0, 1], i \in \mathbb{Z} \quad (6)$$

where $Z_i = \{Z_i(t)\}_{t \in [0, 1]}$, $i \in \mathbb{Z}$ are i.i.d. regularly varying processes and $\Psi_{i,j} = \{\Psi_{i,j}(t)\}_{t \in [0, 1]}$ are random processes. For example, one can consider that at each spatial location t , the temporal dependence between the observations is described by an (SRE) model $X_i(t) = Y_i(t)X_{i-1}(t) + Z_i(t)$, leading to model (6) with $\Psi_{i,0}(t) = 1$ and $\Psi_{i,j}(t) = \prod_{k=1}^j Y_{i-k+1}(t)$ for $j \geq 1$. Our main result shows that, if $(\Psi_{0,j})_{j \geq 0}$ satisfy some moment conditions, and the same “predictability assumption” as in [15] holds for the sequence $\{(\Psi_{0,j}, Z_{-j})\}_{j \geq 0}$, then the series X_0 given by (6) converges a.s. and is regularly varying in \mathbb{D} . We postpone the asymptotic analysis of the normalized maximum of X_1, \dots, X_n for a future study.

The article is organized as follows. In Section 2, we introduce the concept of regular variation on \mathbb{D} and discuss some of its properties. In Section 3, we state and prove our result about the regular variation in \mathbb{D} of a series of the form (6). For this, we use two preliminary results, one of them being a version of Breiman’s lemma for \mathbb{D} -valued random elements. The appendix contains a variant of Pratt’s lemma (regarding the interchanging of lim sup with an integral), which is needed for checking the relative compactness conditions mentioned above.

2 Regular variation on \mathbb{D}

In this section, we recall the definition and main properties of the regular variation for random processes with sample paths in \mathbb{D} . We follow references [18, 13, 8].

We let $\mathbb{D} = \mathbb{D}[0, 1]$ be the space of right continuous functions $x : [0, 1] \rightarrow \mathbb{R}$ with left limits. Recall that \mathbb{D} is a complete separable metric space (CSMS), equipped with a distance called d_0 , which is equivalent to Skorohod J_1 -metric (see pages 109-115 of [4]). We denote by $\mathcal{B}(\mathbb{D})$ the class of Borel sets in \mathbb{D} , equipped with the J_1 -topology. Note that $\|x\|_\infty = \sup_{t \in [0, 1]} |x(t)| < \infty$ for any $x \in \mathbb{D}$, and the topology of uniform convergence on \mathbb{D} is stronger than the J_1 -topology.

We let $\mathbb{S}_{\mathbb{D}} = \{x \in \mathbb{D}; \|x\|_\infty = 1\}$ be the “unit sphere” in \mathbb{D} , endowed with metric d_0 , and $\mathcal{B}(\mathbb{S}_{\mathbb{D}})$ be the class of Borel sets in $\mathbb{S}_{\mathbb{D}}$. We denote $\mathbb{D}_0 = \mathbb{D} \setminus \{0\}$, where 0 is the null function in \mathbb{D} , and let $\mathcal{B}(\mathbb{D}_0)$ be the class of Borel sets in \mathbb{D}_0 . Similarly to the polar coordinate transformation in \mathbb{R}^d , we consider the homeomorphism $T : \mathbb{D}_0 \rightarrow (0, \infty) \times \mathbb{S}_{\mathbb{D}}$ given by $T(x) = (\|x\|_\infty, x/\|x\|_\infty)$. We define the space

$$\overline{\mathbb{D}}_0 := (0, \infty] \times \mathbb{S}_{\mathbb{D}}.$$

This space is endowed with the product metric, where $(0, \infty]$ has the metric $\rho(x, y) = (1/x) - (1/y)$, with the convention $1/\infty = 0$. We let $\mathcal{B}(\overline{\mathbb{D}}_0)$ be the class of all Borel sets in $\overline{\mathbb{D}}_0$. Note that $\overline{\mathbb{D}}_0 \setminus T(\mathbb{D}_0) = \{\infty\} \times \mathbb{S}_{\mathbb{D}}$.

Remark 2.1. The authors of [18, 13, 8] identify the space \mathbb{D} with $[0, \infty) \times \mathbb{S}_{\mathbb{D}}$ and write $\overline{\mathbb{D}}_0 \setminus \mathbb{D} = \{\infty\} \times \mathbb{S}_{\mathbb{D}}$. For the sake of the analogy with \mathbb{R}^d , we prefer to distinguish between \mathbb{D}_0 and $(0, \infty) \times \mathbb{S}_{\mathbb{D}}$.

Similarly to \mathbb{R}^d , the concept of regular variation on \mathbb{D} can be defined using convergence of measures. A small technical issue is the fact that $\overline{\mathbb{D}}_0$ is not a LCCB space, and hence the notion of vague convergence is not appropriate on this space. Fortunately, $\overline{\mathbb{D}}_0$ is a CSMS and vague convergence can be replaced by the \hat{w} -convergence. Recall that a measure μ on a CSMS E (with metric d) is *boundedly finite* if $\mu(B) < \infty$ for any bounded Borel set B in E . (A set B is bounded if it is contained in an open sphere $S_r(x) = \{y \in E; d(x, y) < r\}$.) A sequence $(\mu_n)_n$ of boundedly finite measures converges to a boundedly finite measure μ in the \hat{w} -topology (written as $\mu_n \xrightarrow{\hat{w}} \mu$) if $\mu_n(B) \rightarrow \mu(B)$ for any bounded Borel set B with $\mu(\partial B) = 0$ (see Appendix A2.6 of [7]).

The following definition introduces the analogue of (2) for \mathbb{D} . Let ν_α be the measure on $(0, \infty]$ given by $\nu_\alpha(dx) = \alpha x^{-\alpha-1} 1_{(0, \infty)}(x) dx$, $\nu_\alpha(\{\infty\}) = 0$.

Definition 2.2. A process $Z = \{Z(t)\}_{t \in [0,1]}$ with sample paths in \mathbb{D} is called *regularly varying* if there exist $\alpha > 0, c > 0$, a sequence $(a_n)_{n \geq 1}$ with $a_n > 0, a_n \uparrow \infty$, and a probability measure σ on $\mathbb{S}_{\mathbb{D}}$ such that

$$nP(T(a_n^{-1}Z) \in \cdot) \xrightarrow{\hat{w}} c\nu_{\alpha} \times \sigma \quad \text{in } \overline{\mathbb{D}}_0.$$

α is called the *index* of Z .

Remark 2.3. The previous definition coincides with Definition 2 of [13], if we identify $T(x)$ with x , for any $x \in \mathbb{D}_0$. Note that $\mu = c\nu_{\alpha} \times \sigma$ is a non-null boundedly finite measure on $\overline{\mathbb{D}}_0$ which satisfies $\mu(\{\infty\} \times \mathbb{S}_{\mathbb{D}}) = 0$.

Remark 2.4. We now examine the analogue of (1) for \mathbb{D} . Suppose that Z is regularly varying as in Definition 2.2. Let \mathcal{P} be the class of sets of the form

$$V_{a,b;S} = \{x \in \mathbb{D}_0; a < \|x\|_{\infty} \leq b, x/\|x\|_{\infty} \in S\} = T^{-1}((a,b] \times S),$$

for some $0 < a < b < \infty$ and $S \in \mathcal{B}(\mathbb{S}_{\mathbb{D}})$. Define $\nu(V_{a,b;S}) = c\nu_{\alpha}((a,b])\sigma(S)$. Since \mathcal{P} is a semiring which generates $\mathcal{B}(\mathbb{D}_0)$, by Theorem 11.3 of [5], ν can be extended to a measure on \mathbb{D}_0 . Let $V_{r;S} = \{x \in \mathbb{D}; \|x\|_{\infty} > r, x/\|x\|_{\infty} \in S\}$ be the \mathbb{D} -analogue of a “pizza-slice” set from \mathbb{R}^d , with $0 < r < \infty$. Then

$$\nu(V_{r;S}) = cr^{-\alpha}\sigma(S), \tag{7}$$

and hence $c = \nu(V_{1;\mathbb{S}_{\mathbb{D}}})$. It can be shown that ν satisfies the scaling property $\nu(sB) = s^{-\alpha}\nu(B)$ for any $s > 0$ and $B \in \mathcal{B}(\mathbb{D}_0)$, $\nu(\partial V_{r;S}) = cr^{-\alpha}\sigma(\partial S)$ and

$$nP(a_n^{-1}Z \in V_{r;S}) \rightarrow \nu(V_{r;S}) \tag{8}$$

for any $r > 0$ and $S \in \mathcal{B}(\mathbb{S}_{\mathbb{D}})$ with $\sigma(\partial S) = 0$ (see the proofs of Theorems 1.14 and 1.15 of [18] for \mathbb{R}^d ; the same arguments work for \mathbb{D}). But (8) cannot be expressed as a statement of \hat{w} -convergence, because there is no natural “infinity” that can be added to \mathbb{D}_0 . Taking $S = \mathbb{S}_{\mathbb{D}}$ in (8), we obtain that for any $r > 0$,

$$nP(\|Z\|_{\infty} > a_nr) \rightarrow cr^{-\alpha}. \tag{9}$$

By abuse of notation, we write $Z \in \text{RV}(\{a_n\}, \nu, \overline{\mathbb{D}}_0)$, although ν is a measure on \mathbb{D}_0 , not on $\overline{\mathbb{D}}_0$. Note that

$$\nu \circ T^{-1} = c\nu_{\alpha} \times \sigma \quad \text{on } (0, \infty) \times \mathbb{S}_{\mathbb{D}}.$$

To introduce another characterization of regular variation on \mathbb{D} , we need to recall the definition of the modulus of continuity: for any $x \in \mathbb{D}$ and $\delta > 0$,

$$w(x, \delta) = \sup_{|s-t| \leq \delta} |x(s) - x(t)|.$$

If x is continuous, then $\lim_{\delta \rightarrow 0} w(x, \delta) = 0$. In the case of a discontinuous function $x \in \mathbb{D}$, the following quantity plays the same role as $w(x, \delta)$:

$$w''(x, \delta) = \sup_{t_1 \leq t \leq t_2, t_2 - t_1 \leq \delta} |x(t) - x(t_1)| \wedge |x(t_2) - x(t)|,$$

since $\lim_{\delta \rightarrow 0} w''(x, \delta) = 0$ (see (14.8) and (14.46) of [4]). We define

$$w(x, T) = \sup_{s, t \in T} |x(s) - x(t)| \quad \text{for any set } T \subset [0, 1].$$

The following result will be needed in the sequel. This result shows that the regular variation in \mathbb{D} coincides with the regular variation of the marginal distributions, combined with some relative compactness conditions.

Theorem 2.5 (Theorem 10 of [13]). *Let $Z = \{Z(t)\}_{t \in [0, 1]}$ be a process with sample paths in \mathbb{D} . The following statements are equivalent:*

- (i) $Z \in \text{RV}(\{a_n\}, \nu, \overline{\mathbb{D}}_0)$;
- (ii) *There exists a sequence $(a_n)_{n \geq 1}$ with $a_n > 0, a_n \uparrow \infty$, a set $T \subset [0, 1]$ containing 0, 1 with $[0, 1] \setminus T$ countable, and a collection $\{\nu_{t_1, \dots, t_k}; t_1, \dots, t_k \in T, k \geq 1\}$, each ν_{t_1, \dots, t_k} being a Radon measure on $\overline{\mathbb{R}}_0^k$ with $\nu_{t_1, \dots, t_k}(\overline{\mathbb{R}}_0^k \setminus \mathbb{R}_0^k) = 0$ and ν_t is non-null for some $t \in T$, such that:*
 - (a) $(Z(t_1), \dots, Z(t_k)) \in \text{RV}(\{a_n\}, \nu_{t_1, \dots, t_k}, \overline{\mathbb{R}}_0^k)$ for all $t_1, \dots, t_k \in T$; and
 - (b) *the following three conditions are satisfied:*

$$(C1) \quad \lim_{\delta \rightarrow 0} \limsup_{n \rightarrow \infty} nP(w''(Z, \delta) > a_n \varepsilon) = 0 \text{ for any } \varepsilon > 0;$$

$$(C2) \quad \lim_{\delta \rightarrow 0} \limsup_{n \rightarrow \infty} nP(w(Z, [0, \delta]) > a_n \varepsilon) = 0 \text{ for any } \varepsilon > 0;$$

$$(C3) \quad \lim_{\delta \rightarrow 0} \limsup_{n \rightarrow \infty} nP(w(Z, [1 - \delta, 1]) > a_n \varepsilon) = 0 \text{ for any } \varepsilon > 0.$$

Remark 2.6. The sequences $\{a_n\}$ in (i) and (ii) can be taken to be the same. The measure ν is uniquely determined by $\{\nu_{t_1, \dots, t_k}; t_1, \dots, t_k \in T, k \geq 1\}$ and

$$\nu_{t_1, \dots, t_k}(B) = \nu(\pi_{t_1, \dots, t_k}^{-1}(B \cap \mathbb{R}^k)), \quad \text{for all } B \in \mathcal{B}(\overline{\mathbb{R}}_0^k) \quad (10)$$

where $\pi_{t_1, \dots, t_k}(x) = (x(t_1), \dots, x(t_k))$, $x \in \mathbb{D}$. The set T in (ii) can be taken to be the set of all $t \in [0, 1]$ such that $\nu(\{x \in \mathbb{D}_0; x \text{ is not continuous at } t\}) = 0$.

3 The main result

We are now ready to state our main result.

Theorem 3.1. *Let $Z_j = \{Z_j(t)\}_{t \in [0,1]}$, $j \geq 1$ be i.i.d. processes with sample paths in \mathbb{D} such that $Z_1 \in \text{RV}(\{a_n\}, \nu, \overline{\mathbb{D}}_0)$, and $\alpha > 0$ be the index of Z_1 . Let $\Psi_j = \{\Psi_j(t)\}_{t \in [0,1]}$, $j \geq 1$ be some processes with continuous sample paths, such that $P(\cup_{j \geq 1} \{\|\Psi_j\|_\infty > 0\}) = 1$, and there exists an $m \geq 1$ and a set $T_1 \subset [0, 1]$ containing 0 and 1, with $[0, 1] \setminus T_1$ countable, for which*

$$P\left(\bigcup_{j=1}^m \{\Psi_j(t) \neq 0\}\right) > 0 \quad \text{for all } t \in T_1. \quad (11)$$

Suppose that $(\Psi_1, \dots, \Psi_j, Z_1, \dots, Z_{j-1})$ is independent of $(Z_k)_{k \geq j}$ for any $j \geq 2$, Ψ_1 is independent of $(Z_j)_{j \geq 1}$, and there exists $\gamma \in (0, \alpha)$ such that:

$$\begin{aligned} \sum_{j=1}^m E\|\Psi_j\|_\infty^{\alpha-\gamma} < \infty \text{ and } \sum_{j \geq 1} E\|\Psi_j\|_\infty^{\alpha+\gamma} < \infty \quad & \text{if } \alpha \in (0, 1) \cup (1, 2), \\ E\left(\sum_{j \geq 1} \|\Psi_j\|_\infty^{\alpha-\gamma}\right)^{(\alpha+\gamma)/(\alpha-\gamma)} < \infty \quad & \text{if } \alpha \in \{1, 2\}, \text{ or} \\ E\left(\sum_{j \geq 1} \|\Psi_j\|_\infty^2\right)^{(\alpha+\gamma)/2} < \infty \quad & \text{if } \alpha > 2. \end{aligned}$$

Then the series $X = \sum_{j \geq 1} \Psi_j Z_j$ converges in \mathbb{D} a.s. and $X \in \text{RV}(\{a_n\}, \nu^X, \overline{\mathbb{D}}_0)$ where

$$\nu^X(\cdot) = E \left[\sum_{j \geq 1} \nu \circ h_{\Psi_j}^{-1}(\cdot) \right].$$

For any $\psi \in \mathbb{D}$, we define the product map $h_\psi : \mathbb{D} \rightarrow \mathbb{D}$ by $h_\psi(x) = \psi x$, $x \in \mathbb{D}$, with $(\psi x)(t) = \psi(t)x(t)$ for any $t \in [0, 1]$.

We begin with some preliminary results. The following result is known in the literature as *Breiman's lemma* (see [6]).

Lemma 3.2. *Let Z and Y be independent nonnegative random variables such that $Z \in \text{RV}(\{a_n\}, \nu, \overline{\mathbb{R}}_0)$ and $0 < E(Y^{\alpha+\gamma}) < \infty$ for some $\gamma > 0$, where $\alpha > 0$ is the index of Z (and hence, $\nu(r, \infty) = cr^{-\alpha}$ for any $r > 0$, for some $c > 0$). Then $X = YZ \in \text{RV}(\{a_n\}, \nu^X, \overline{\mathbb{R}}_0)$ where $\nu^X(r, \infty) = cr^{-\alpha}E(Y^\alpha)$ for any $r > 0$.*

Note that in Breiman's lemma, $\nu^X(\cdot) = E[\nu \circ h_Y^{-1}(\cdot \cap [0, \infty))]$, where $h_Y(x) = yx$ for any $x, y \in [0, \infty)$.

Proposition A.1 of [3] gives an extension of Lemma 3.2 to a product $X = AZ$, where Z is regularly varying in \mathbb{R}^d (with index α), A is a random matrix with $0 < E\|A\|^{\alpha+\gamma} < \infty$ for some $\gamma > 0$, and Z, A are independent. Lemma 4.3 of [15] extends this result to a finite sum $X = \sum_{j=1}^m A_j Z_j$, where $(Z_j)_j$ are i.i.d. regularly varying in \mathbb{R}^d , $(A_j)_j$ are random matrices with $E\|A_j\|^{\alpha+\gamma} < \infty$ for some $\gamma > 0$, and Z_j is independent of $(A_1, \dots, A_j, Z_1, \dots, Z_{j-1})$ for all j . (For the later result, one also needs the hypothesis $P(\cup_{j=1}^m \{\|A_j\| > 0\}) > 0$, which is missing from [15].)

Our first result is a version of Breiman's lemma for processes with sample paths in \mathbb{D} .

Lemma 3.3. *Let $Z = \{Z(t)\}_{t \in [0,1]}$ and $\Psi = \{\Psi(t)\}_{t \in [0,1]}$ be independent processes with sample paths in \mathbb{D} such that $Z \in \text{RV}(\{a_n\}, \nu, \overline{\mathbb{D}}_0)$, Ψ has continuous sample paths, and $E\|\Psi\|_\infty^{\alpha+\gamma} < \infty$ for some $\gamma > 0$, where $\alpha > 0$ is the index of Z . Suppose that there exists a set $T_1 \subset [0, 1]$ containing 0 and 1 with $[0, 1] \setminus T_1$ countable, such that*

$$P(\Psi(t) \neq 0) > 0 \quad \text{for all } t \in T_1. \quad (12)$$

Then $X = \Psi Z \in \text{RV}(\{a_n\}, \nu^X, \overline{\mathbb{D}}_0)$ where

$$\nu^X(\cdot) = E[\nu \circ h_\Psi^{-1}(\cdot)]. \quad (13)$$

Proof: Let $T \subset [0, 1]$ and $\{\nu_{t_1, \dots, t_k}; t_1, \dots, t_k \in T, k \geq 1\}$ be the set and the marginal measures given by Theorem 2.5.(ii) for Z_1 . We show that X satisfies the conditions of Theorem 2.5.(ii) with $T_X = T \cap T_1$ instead of T and the measures ν_{t_1, \dots, t_k}^X (defined by (14) below) instead of ν_{t_1, \dots, t_k} .

First we show that X satisfies condition (a). For this, let $t_1, \dots, t_k \in T_X$ be arbitrary. Note that $(X(t_1), \dots, X(t_k))^T = AY$ where A is the diagonal matrix with entries $\Psi(t_1), \dots, \Psi(t_k)$ and $Y = (Z(t_1), \dots, Z(t_k))^T$. By Proposition A.1 of [3], $(X(t_1), \dots, X(t_k)) \in \text{RV}(\{a_n\}, \nu_{t_1, \dots, t_k}^X, \overline{\mathbb{R}}_0^k)$, where

$$\nu_{t_1, \dots, t_k}^X(\cdot) = E[\nu_{t_1, \dots, t_k} \circ h_A^{-1}(\cdot \cap \mathbb{R}^k)] \quad (14)$$

and $h_A : \mathbb{R}^k \rightarrow \mathbb{R}^k$ is given by $h_A(x) = Ax$. To justify the application of this proposition, we note that $E\|A\|^{\alpha+\gamma} \leq E\|\Psi\|_\infty^{\alpha+\gamma} < \infty$ and $E\|A\|^{\alpha+\gamma} > 0$, where $\|A\| = \max_{i \leq k} |\Psi(t_i)|$. (If $E\|A\|^{\alpha+\gamma} = 0$ then $P(\Psi(t_i) = 0) = 1$ for all $i \leq k$, which contradicts (12).)

We now prove that the measure ν^X given by (13) has marginal measures ν_{t_1, \dots, t_k}^X . For any $B \in \mathcal{B}(\overline{\mathbb{R}}_0^k)$, we have

$$\begin{aligned}
\nu_{t_1, \dots, t_k}^X(B) &= E[\nu_{t_1, \dots, t_k}\{x \in \mathbb{R}^k; Ax \in B \cap \mathbb{R}^k\}] \\
&= E[\nu_{t_1, \dots, t_k}\{x \in \mathbb{R}^k; (\Psi(t_1)x_1, \dots, \Psi(t_k)x_k) \in B \cap \mathbb{R}^k\}] \\
&= E[\nu\{x \in \mathbb{D}; (\Psi(t_1)x(t_1), \dots, \Psi(t_k)x(t_k)) \in B \cap \mathbb{R}^k\}] \\
&= E[\nu\{x \in \mathbb{D}; h_\Psi(x) \in \{y \in \mathbb{D}; (y(t_1), \dots, y(t_k)) \in B \cap \mathbb{R}^k\}\}] \\
&= \nu^X\{y \in \mathbb{D}; (y(t_1), \dots, y(t_k)) \in B \cap \mathbb{R}^k\} \\
&= \nu^X(\pi_{t_1, \dots, t_k}^{-1}(B \cap \mathbb{R}^k))
\end{aligned}$$

using (10) for the third equality and (13) for the second last equality.

Next we show that X satisfies condition (b). We only prove (C1). Conditions (C2) and (C3) can be proved similarly. Let $\varepsilon > 0$ be arbitrary. If $w''(X, \delta) > a_n \varepsilon$ then there exist some $t_1 \leq t \leq t_2$ with $t_2 - t_1 \leq \delta$ such that $|X(t) - X(t_1)| > a_n \varepsilon$ and $|X(t_2) - X(t)| > a_n \varepsilon$. Since

$$\begin{aligned}
|X(t) - X(t_1)| &\leq |Z(t)| |\Psi(t) - \Psi(t_1)| + |\Psi(t_1)| |Z(t) - Z(t_1)| \\
&\leq \|Z\|_\infty |\Psi(t) - \Psi(t_1)| + \|\Psi\|_\infty |Z(t) - Z(t_1)|,
\end{aligned}$$

it follows that $\|Z\|_\infty |\Psi(t) - \Psi(t_1)| > a_n \varepsilon / 2$ or $\|\Psi\|_\infty |Z(t) - Z(t_1)| > a_n \varepsilon / 2$. Similarly, $\|Z\|_\infty |\Psi(t_2) - \Psi(t)| > a_n \varepsilon / 2$ or $\|\Psi\|_\infty |Z(t_2) - Z(t)| > a_n \varepsilon / 2$. Hence

$$\begin{aligned}
nP(w''(X, \delta) > a_n \varepsilon) &\leq nP(\|Z\|_\infty w''(\Psi, \delta) > a_n \varepsilon / 2) + nP(\|\Psi\|_\infty w''(Z, \delta) > a_n \varepsilon / 2) \\
&\quad + 2nP(\|Z\|_\infty w(\Psi, \delta) > a_n \varepsilon / 2) \\
&=: P_{n,1}(\delta) + P_{n,2}(\delta) + P_{n,3}(\delta).
\end{aligned}$$

We treat separately the three terms.

For the first term, we note that for any $\theta > 0$,

$$P_{n,1}(\delta) \leq nP(\|Z\|_\infty \theta > a_n \varepsilon / 2) + nP(\|Z\|_\infty w''(\Psi, \delta) 1_{\{w''(\Psi, \delta) > \theta\}} > a_n \varepsilon / 2).$$

We take the limit as $n \rightarrow \infty$. Using (9) for the first term and Lemma 3.2 for the second term, we obtain that, for any $\theta > 0$,

$$\limsup_{n \rightarrow \infty} P_{n,1}(\delta) \leq c(\varepsilon/2)^{-\alpha} \theta^\alpha + c(\varepsilon/2)^{-\alpha} E[w''(\Psi, \delta)^\alpha 1_{\{w''(\Psi, \delta) > \theta\}}].$$

Taking $\theta \rightarrow 0$, we obtain that $\limsup_{n \rightarrow \infty} P_{n,1}(\delta) \leq c(\varepsilon/2)^{-\alpha} E[w''(\Psi, \delta)^\alpha]$.

Take the limit as $\delta \rightarrow 0$. By the dominated convergence theorem, $\lim_{\delta \rightarrow 0} E[w''(\Psi, \delta)^\alpha] = 0$, since $\lim_{\delta \rightarrow 0} w''(\Psi, \delta) = 0$ and $w''(\Psi, \delta) \leq 2\|\Psi\|_\infty$. Hence

$$\lim_{\delta \rightarrow 0} \limsup_{n \rightarrow \infty} P_{n,1}(\delta) = 0.$$

For the second term, we denote by P_Ψ the law of Ψ on \mathbb{D} . Since Z and Ψ are independent, we have:

$$P_{n,2}(\delta) = \int_{\mathbb{D}} nP(\|\psi\|_\infty w''(Z, \delta) > a_n \varepsilon/2) P_\Psi(d\psi).$$

Using Lemma A.1 (Appendix A), we infer that:

$$\limsup_{n \rightarrow \infty} P_{n,2}(\delta) \leq \int_{\mathbb{D}} \limsup_{n \rightarrow \infty} nP(\|\psi\|_\infty w''(Z, \delta) > a_n \varepsilon/2) P_\Psi(d\psi). \quad (15)$$

To justify the application of this lemma, we note that

$$f_n(\psi) := nP(\|\psi\|_\infty w''(Z, \delta) > a_n \varepsilon/2) \leq g_n(\psi) := nP(\|\psi\|_\infty \|Z\|_\infty > a_n \varepsilon/4),$$

$$g_n(\psi) \rightarrow g(\psi) := c(\varepsilon/4)^{-\alpha} \|\psi\|_\infty^\alpha \text{ (due to (9)) and}$$

$$\int_{\mathbb{D}} g_n(\psi) P_\Psi(d\psi) = nP(\|\Psi\|_\infty \|Z\|_\infty > a_n \varepsilon/4) \rightarrow c(\varepsilon/4)^{-\alpha} E\|\Psi\|_\infty^\alpha = \int_{\mathbb{D}} g(\psi) P_\Psi(d\psi),$$

by Lemma 3.2. (Note that $\|Z\|_\infty$ is regularly varying, and (12) implies that $P(\|\Psi\|_\infty > 0) > 0$, which forces $E\|\Psi\|_\infty^{\alpha+\gamma} > 0$.)

We now take the limit as $\delta \rightarrow 0$ in (15). We apply again Lemma A.1 to interchange the limit with the integral. (Both terms are increasing functions of δ , so the limit as $\delta \rightarrow 0$ exists.) Since Z is regularly varying, $\lim_{\delta \rightarrow 0} \limsup_n nP(\|\psi\|_\infty w''(Z, \delta) > a_n \varepsilon/2) = 0$ for any $\psi \in \mathbb{D}$, and hence

$$\lim_{\delta \rightarrow 0} \limsup_{n \rightarrow \infty} P_{n,2}(\delta) = 0.$$

This second application of Lemma A.1 is justified by the fact that $F_\delta(\psi) := \limsup_n nP(\|\psi\|_\infty w''(Z, \delta) > a_n \varepsilon/2) \leq \limsup_n nP(\|\psi\|_\infty \|Z\|_\infty > a_n \varepsilon/4) := G(\psi)$ which does not depend on δ .

It remains to treat the third term. Since Ψ has continuous sample paths, $\lim_{\delta \rightarrow 0} w(\Psi, \delta) = 0$, and this term is treated exactly as the first term. \square

We now consider a finite sum of product terms as in the previous lemma.

Lemma 3.4. *Let $Z_j = \{Z_j(t)\}_{t \in [0,1]}$, $j = 1, \dots, m$ be i.i.d. processes with sample paths in \mathbb{D} such that $Z_1 \in \text{RV}(\{a_n\}, \nu, \overline{\mathbb{D}}_0)$ and $\alpha > 0$ be the index of Z_1 . Let $\Psi_j = \{\Psi_j(t)\}_{t \in [0,1]}$, $j = 1, \dots, m$ be some processes with continuous sample paths such that Ψ_1 is independent of Z_1 and $(\Psi_1, \dots, \Psi_j, Z_1, \dots, Z_{j-1})$*

is independent of Z_j for any $j = 2, \dots, m$. Suppose that there exists a set $T_1 \subset [0, 1]$ containing 0 and 1 with $[0, 1] \setminus T_1$ countable, such that (11) holds, and there exists $\gamma > 0$ such that $E\|\Psi_j\|_\infty^{\alpha+\gamma} < \infty$ for all $j = 1, \dots, m$. Then $X = \sum_{j=1}^m \Psi_j Z_j \in \text{RV}(\{a_n\}, \nu^X, \overline{\mathbb{D}}_0)$ where

$$\nu^X = E\left[\sum_{j=1}^m \nu \circ h_{\Psi_j}^{-1}(\cdot \cap \mathbb{D})\right]. \quad (16)$$

Proof: Let $T \subset [0, 1]$ and $\{\nu_{t_1, \dots, t_k}; t_1, \dots, t_k \in T, k \geq 1\}$ be the set and the marginal measures given by Theorem 2.5.(ii) for Z_1 . We show that X satisfies the conditions of Theorem 2.5.(ii) with $T_X = T \cap T_1$ instead of T and the measures ν_{t_1, \dots, t_k}^X (defined by (17) below) instead of ν_{t_1, \dots, t_k} .

First we show that X satisfies condition (a). For this, let $t_1, \dots, t_k \in T_X$ be arbitrary. Note that $(X(t_1), \dots, X(t_k))^T = \sum_{j=1}^m A_j Y_j$ where A_j is the diagonal matrix with entries $\Psi_j(t_1), \dots, \Psi_j(t_k)$ and $Y_j = (Z_j(t_1), \dots, Z_j(t_k))^T$. By Lemma 4.3 of [15], $(X(t_1), \dots, X(t_k)) \in \text{RV}(\{a_n\}, \nu_{t_1, \dots, t_k}^X, \overline{\mathbb{R}}_0^k)$ where

$$\nu_{t_1, \dots, t_k}^X = E\left[\sum_{j=1}^m \nu \circ h_{A_j}^{-1}(\cdot \cap \mathbb{R}^k)\right]. \quad (17)$$

To justify the application of this lemma, we note that $E\|A_j\|^{\alpha+\gamma} \leq E\|\Psi_j\|_\infty^{\alpha+\gamma} < \infty$ for any j , and $P(\cup_{j=1}^m \{\|A_j\| > 0\}) > 0$, where $\|A_j\| = \max_{i \leq k} |\Psi_j(t_i)|$. (If $P(\cap_{j=1}^m \{\|A_j\| = 0\}) = 1$ then $P(\cap_{j=1}^m \{\Psi_j(t_i) = 0\}) = 1$ for any $i = 1, \dots, k$, which contradicts (11).) The fact that the measure ν^X given by (16) has the marginal measures ν_{t_1, \dots, t_k}^X follows as in the case $m = 1$ (see the proof of Lemma 3.3). We omit the details.

Next we show that X satisfies condition (b). We only prove (C1). Conditions (C2) and (C3) can be proved by similar methods.

To simplify the notation, we assume that $m = 2$. The general result can be proved similarly. Let $\varepsilon > 0$ and $\theta > 0$ be arbitrary. As in the proof of Lemma 5.1 of [8], we use the decomposition:

$$\begin{aligned} nP(w''(X, \delta) > a_n \varepsilon) &= nP(w''(X, \delta) > a_n \varepsilon, \|\Psi_1 Z_1\|_\infty > a_n \theta, \|\Psi_2 Z_2\|_\infty > a_n \theta) \\ &+ nP(w''(X, \delta) > a_n \varepsilon, \|\Psi_1 Z_1\|_\infty > a_n \theta, \|\Psi_2 Z_2\|_\infty \leq a_n \theta) \\ &+ nP(w''(X, \delta) > a_n \varepsilon, \|\Psi_1 Z_1\|_\infty \leq a_n \theta, \|\Psi_2 Z_2\|_\infty > a_n \theta) \\ &+ nP(w''(X, \delta) > a_n \varepsilon, \|\Psi_1 Z_1\|_\infty \leq a_n \theta, \|\Psi_2 Z_2\|_\infty \leq a_n \theta) \\ &:= P_{n,1}(\delta) + P_{n,2}(\delta) + P_{n,3}(\delta) + P_{n,4}(\delta). \end{aligned}$$

We treat separately the four terms. For the first term we use the fact that $\|xy\|_\infty \leq \|x\|_\infty \|y\|_\infty$ for all $x, y \in \mathbb{D}$. We denote by P_{Ψ_1, Ψ_2, Z_1} the law of (Ψ_1, Ψ_2, Z_1) . Using the independence between (Ψ_1, Ψ_2, Z_1) and Z_2 , we have:

$$\begin{aligned} P_{n,1}(\delta) &\leq nP(\|\Psi_1\|_\infty \|Z_1\|_\infty > a_n\theta, \|\Psi_2\|_\infty \|Z_2\|_\infty > a_n\theta) \\ &= \int_{\mathbb{D}^3} f_n(\psi_1, \psi_2, z_1) dP_{\Psi_1, \Psi_2, Z_1}(\psi_1, \psi_2, z_1). \end{aligned}$$

where $f_n(\psi_1, \psi_2, z_1) = n1_{\{\|\psi_1\|_\infty \|z_1\|_\infty > a_n\theta\}} P(\|\psi_2\|_\infty \|Z_2\|_\infty > a_n\theta) \rightarrow 0$. By Lemma A.1 (Appendix A), it follows that for any $\delta > 0$

$$\limsup_{n \rightarrow \infty} P_{n,1}(\delta) \leq \int_{\mathbb{D}^3} \limsup_{n \rightarrow \infty} f_n(\psi_1, \psi_2, z_1) dP_{\Psi_1, \Psi_2, Z_1}(\psi_1, \psi_2, z_1) = 0.$$

To justify the application of this lemma, we note that $f_n \leq g_n$ where $g_n(\psi_1, \psi_2, z_1) = nP(\|\psi_2\|_\infty \|Z_2\|_\infty > a_n\theta) \rightarrow g(\psi_1, \psi_2, z_1) = c\theta^{-\alpha} \|\psi_2\|_\infty^\alpha$, and

$$\int_{\mathbb{D}^3} g_n dP_{\Psi_1, \Psi_2, Z_1} = nP(\|\Psi_2\|_\infty \|Z_2\|_\infty > a_n\theta) \rightarrow c\theta^{-\alpha} E\|\Psi_2\|_\infty^\alpha = \int_{\mathbb{D}^3} g dP_{\Psi_1, \Psi_2, Z_1}.$$

(The last convergence follows by Lemma 3.2 if $P(\|\Psi_2\|_\infty > 0) > 0$, and holds trivially if $\|\Psi_2\|_\infty = 0$ a.s.)

For the second term, we use the fact that $w''(x+y, \delta) \leq w''(x, \delta) + 2\|y\|_\infty$ for any $x, y \in \mathbb{D}$. Hence,

$$\begin{aligned} P_{n,2}(\delta) &\leq nP(w''(\Psi_1 Z_1, \delta) + 2\|\Psi_2 Z_2\|_\infty > a_n\varepsilon, \|\Psi_2 Z_2\|_\infty \leq a_n\theta) \\ &\leq nP(w''(\Psi_1 Z_1, \delta) > a_n(\varepsilon - 2\theta)). \end{aligned}$$

Therefore, if $\theta < \varepsilon/2$, then by Lemma 3.3,

$$\lim_{\delta \rightarrow 0} \limsup_{n \rightarrow \infty} P_{n,2}(\delta) \leq \lim_{\delta \rightarrow 0} \limsup_{n \rightarrow \infty} nP(w''(\Psi_1 Z_1, \delta) > a_n(\varepsilon - 2\theta)) = 0.$$

The third term is similar to the second term. For the fourth term, we use the fact that $w''(x+y, \delta) \leq 2\|x+y\|_\infty \leq 2(\|x\|_\infty + \|y\|_\infty)$ and hence,

$$P_{n,4}(\delta) \leq nP(\|\Psi_1 Z_1\|_\infty + \|\Psi_2 Z_2\|_\infty > a_n\varepsilon/2, \|\Psi_1 Z_1\|_\infty \leq a_n\theta, \|\Psi_2 Z_2\|_\infty \leq a_n\theta).$$

The last probability is 0 if $\theta < \varepsilon/4$. The conclusion follows. \square

Proof of Theorem 3.1: *Step 1.* We show that with probability 1, the series $X(t)$ converges for any $t \in [0, 1]$, and the process $X = \{X(t)\}_{t \in [0, 1]}$ has sample paths in \mathbb{D} . By applying Theorem 3.1 of [15] to the regularly varying random variables $\{\|Z_j\|_\infty\}_{j \geq 1}$, the random coefficients $\{\|\Psi_j\|_\infty\}_{j \geq 1}$, and the filtration $\{\mathcal{F}_j\}_{j \geq 1}$ given by: $\mathcal{F}_1 = \sigma(\Psi_1)$ and $\mathcal{F}_j = \sigma(\Psi_1, \dots, \Psi_j, Z_1, \dots, Z_{j-1})$ for $j \geq 2$, we infer that $\sum_{j \geq 1} \|\Psi_j\|_\infty \|Z_j\|_\infty < \infty$ a.s. Hence, with probability 1, for any $t \in [0, 1]$,

$$|X(t)| \leq \sum_{j \geq 1} |\Psi_j(t)| |Z_j(t)| \leq \sum_{j \geq 1} \|\Psi_j\|_\infty \|Z_j\|_\infty < \infty.$$

Let $X^{(m)} = \sum_{j=1}^m \Psi_j Z_j$. For any $t \in [0, 1]$,

$$|(X^{(m)} - X)(t)| \leq \sum_{j \geq m+1} |\Psi_j(t) Z_j(t)| \leq \sum_{j \geq m+1} \|\Psi_j\|_\infty \|Z_j\|_\infty$$

and hence $\|X^{(m)} - X\|_\infty \leq \sum_{j \geq m+1} \|\Psi_j\|_\infty \|Z_j\|_\infty \rightarrow 0$ a.s. Since the uniform limit of a sequence of functions in \mathbb{D} is in \mathbb{D} , $X \in \mathbb{D}$ a.s. Since uniform convergence implies J_1 -convergence, $d_0(X^{(m)}, X) \rightarrow 0$ a.s.

Step 2. We show that $X \in \text{RV}(\{a_n\}, \nu^X, \overline{\mathbb{D}}_0)$, i.e. $nP(T(a_n^{-1}X) \in \cdot) \xrightarrow{\hat{w}} \mu^X$ in $\overline{\mathbb{D}}_0$, where $\mu^X = \nu^X \circ T^{-1}$ on $(0, \infty) \times \mathbb{S}_{\mathbb{D}}$ and $\mu^X(\{\infty\} \times \mathbb{S}_{\mathbb{D}}) = 0$. By Proposition A2.6.II of [7], this is equivalent to showing that:

$$\lim_{n \rightarrow \infty} nE[f(T(a_n^{-1}X))] = \int_{\overline{\mathbb{D}}_0} f(u) \mu^X(du),$$

for any bounded continuous function $f : \overline{\mathbb{D}}_0 \rightarrow \mathbb{R}$ which vanishes outside a bounded set. Let f be such a function. Suppose that f vanishes outside a set $(r, \infty] \times \mathbb{S}_{\mathbb{D}}$ for some $r > 0$, and $|f(u)| \leq K$ for all $u \in \overline{\mathbb{D}}_0$.

By Lemma 3.4, we know that for m large enough (for which (11) holds),

$$\lim_{n \rightarrow \infty} nE[f(T(a_n^{-1}X^{(m)}))] = \int_{\overline{\mathbb{D}}_0} f(u) \mu^{(m)}(du)$$

where $\mu^{(m)} = \nu^{(m)} \circ T^{-1}$ on $(0, \infty) \times \mathbb{S}_{\mathbb{D}}$, with $\nu^{(m)}(\cdot) = E[\sum_{j=1}^m \nu \circ h_{\Psi_j}^{-1}(\cdot)]$, and $\mu^{(m)}(\{\infty\} \times \mathbb{S}_{\mathbb{D}}) = 0$. We claim that

$$\lim_{m \rightarrow \infty} \int_{\overline{\mathbb{D}}_0} f(u) \mu^{(m)}(du) = \int_{\overline{\mathbb{D}}_0} f(u) \mu^X(du).$$

To see this, note that for any bounded measurable function $g : \mathbb{D}_0 \rightarrow \mathbb{R}$,

$$\int_{\mathbb{D}_0} g(x) \nu^X(dx) = \sum_{j \geq 1} \int_{\Omega} \int_{\mathbb{D}_0} g(\Psi_j(\omega)y) \nu(dy) P(d\omega).$$

Hence

$$\int_{\mathbb{D}_0} f(u) \mu^X(du) = \int_{\mathbb{D}_0} f(T(x)) \nu^X(dx) = \sum_{j \geq 1} \int_{\Omega} \int_{\mathbb{D}_0} f(T(\Psi_j(\omega)y)) \nu(dy) P(d\omega).$$

Similarly,

$$\int_{\mathbb{D}_0} f(u) \mu^{(m)}(du) = \int_{\mathbb{D}_0} f(T(x)) \nu^{(m)}(dx) = \sum_{j=1}^m \int_{\Omega} \int_{\mathbb{D}_0} f(T(\Psi_j(\omega)y)) \nu(dy) P(d\omega),$$

and therefore,

$$\begin{aligned} & \left| \int_{\mathbb{D}_0} f(u) \mu^{(m)}(du) - \int_{\mathbb{D}_0} f(u) \mu^X(du) \right| \leq \sum_{j \geq m+1} \int_{\Omega} \int_{\mathbb{D}_0} |f(T(\Psi_j(\omega)y))| \nu(dy) P(d\omega) \\ & \leq K \sum_{j \geq m+1} \int_{\Omega} \nu(\{x \in \mathbb{D}_0; \|\Psi_j(\omega)\|_{\infty} \|x\|_{\infty} > r\}) P(d\omega) \\ & = K \sum_{j \geq m+1} \int_{\Omega} c \left(\frac{r}{\|\Psi_j(\omega)\|_{\infty}} \right)^{-\alpha} 1_{\{\|\Psi_j(\omega)\|_{\infty} > 0\}} P(d\omega) \\ & = K c r^{-\alpha} \sum_{j \geq m+1} E \|\Psi_j\|_{\infty}^{\alpha} \rightarrow 0 \quad \text{as } m \rightarrow \infty, \end{aligned}$$

using (7) for the first equality above. It remains to prove that:

$$\lim_{m \rightarrow \infty} \limsup_{n \rightarrow \infty} n E |f(T(a_n^{-1}X)) - f(T(a_n^{-1}X^{(m)}))| = 0.$$

This can be proved similarly to (5.3) of [8], using Lemma 3.4 above and Theorem 3.1 of [15]. We omit the details. \square

A Interchanging limsup with an integral

Pratt's lemma is a useful tool which allows interchanging a limit with an integral (see [20]). In the present article, we need the following version of Pratt's lemma.

Lemma A.1. Let $(f_n)_{n \geq 1}$ and $(g_n)_{n \geq 1}$ be some measurable functions defined on a measure space (E, \mathcal{E}, μ) such that $0 \leq f_n \leq g_n$ for any n , $g_n \rightarrow g$ and

$$\int_E g_n d\mu \rightarrow \int_E g d\mu < \infty. \quad (18)$$

Then

$$\limsup_{n \rightarrow \infty} \int_E f_n d\mu \leq \int_E \limsup_{n \rightarrow \infty} f_n d\mu.$$

Proof: By Fatou's lemma, $\int \liminf_n (g_n - f_n) d\mu \leq \liminf_n \int (g_n - f_n) d\mu$. By (18), g_n (and f_n) are integrable for n large enough, and $\limsup_n f_n$ is also integrable (being bounded by g). Hence, the previous inequality becomes:

$$\int_E g d\mu - \int_E \limsup_n f_n d\mu \leq \liminf_n \int_E g_n d\mu - \limsup_n \int_E f_n d\mu.$$

The conclusion follows by (18). \square

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